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The common asymptotic nature of methods of solving problems of the theory of elasticity for slabs and plates☆

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Abstract

Four approaches to the analytical solution of the three-dimensional problem of the theory of elasticity for slabs and plates are given. All the proposed schemes have a common basis. It is proved that the asymptotic method is identical with the method of hypotheses and the method of successive approximations.

Among the numerous ways of constructing models of the deformation of slabs, plates and binding layers we distinguish four analytical methods: the asymptotic method, the power-series method, the method of hypotheses and the method of successive approximations. When solving the equations of the theory of elasticity these methods are usually applied using non-identical algorithms, and hence the models constructed lead to results which sometimes differ considerably.

Below we propose an algorithm of asymptotic integration, which may form a basis for constructing the method of hypotheses and the method of successive approximations. The solutions of the problem of the theory of elasticity is presented in the form of the sum of two components and is determined by two independent recurrence processes. As a result, the components of the displacement vector and the stress tensor can be expanded in power series, which have the same order for any approximation for all the required quantities.

1. Formulation of the problem

We will consider the equations of the theory of elasticity in Cartesian coordinates x, y, z. We will represent the system of resolvents as follows:

$$\partial_x \sigma_x + \partial_y \tau_{xy} + \partial_z \tau_{xz} = 0 \ (x, y, z) \tag{1.1}$$

$$\partial_x u = \frac{1}{E_x}\sigma_x - \frac{\nu_{xy}}{E_y}\sigma_y - \frac{\nu_{xz}}{E_z}\sigma_z \ (u, \upsilon, w; x, y, z) \tag{1.2}$$

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$$G_{xy}[\partial_x v + \partial_y u] = \tau_{xy}\ (u, v, w;\ x, y, z) \tag{1.3}$$

Relations (1.1) are the equations of equilibrium of an infinitesimal element, and (1.2) and (1.3) are Hooke's law, in which the strains are expressed in terms of the components of the displacement using the Cauchy relations. Here σ_x, $\tau_{xy} = \tau_{yx}$ (x, y, z) are the components of the stress tensor, u, v, w are the components of the displacement vector of an arbitrary point, $G_{xy} = G_{yx}$ (x, y, z) are the shear moduli of the material and $\partial x = \partial/\partial x$ (x, y, z). The moduli of elasticity and Poisson's ratios are related by the expression $\nu_{xy}E_x = \nu_{yx}E_y$ (x, y, z).

Suppose the origin of the system of coordinates lies in the middle plane of the slab, and the z axis is perpendicular to this plane. Assuming that the slab has constant thickness $2h$, henceforth, when necessary, we will use the dimensionless coordinate $\varsigma = z/h$.

2. An algorithm for the asymptotic integration of the equations of the theory of elasticity

We will mean by the asymptotic method of solving the problem of the theory of elasticity the expansion of the stress-strain state in series in a small parameter. We will use half the thickness of the slab h as this parameter.

The main problem of the asymptotic method is to choose the form of the asymptotic series. When solving the system of equations of the theory of elasticity this problem can be solved in different ways.[1–3] We will propose the following version here. We will assume that the stress-strain state of the slab has two components, which we will henceforth denote by superscripts 1 and 2. We will represent the solution of Eqs. (1.1)–(1.3) in the form

$$u = u^{(1)} + u^{(2)}\ (u, v, w); \quad \sigma_x = \sigma_x^{(1)} + \sigma_x^{(2)}, \quad \tau_{xy} = \tau_{xy}^{(1)} + \tau_{xy}^{(2)}\ (x, y, z) \tag{2.1}$$

We define the quantities $u^{(1)}$, $v^{(1)}$, ..., $\tau_{yz}^{(1)}$ and $u^{(2)}$, $v^{(2)}$, ..., $\tau_{yz}^{(2)}$ by two independent recurrence processes, which begin with the expansion of the parameters of the stress-strain state of the slab in asymptotic series. We will take the following expressions for the components of the displacement vector and the stress tensor of the first component

$$\begin{gathered} u^{(1)} = \sum_s h^{s-1} u_1^s\ (u, v), \quad w^{(1)} = \sum_s h^{s-2} w_1^s \\ \sigma_x^{(1)} = \sum_s h^{s-1} \sigma_{x1}^s\ (x, y, z), \quad \tau_{xy}^{(1)} = \sum_s h^{s-3} \tau_{xy1}^s, \quad \tau_{xz}^{(1)} = \sum_s h^{s-2} \tau_{xz1}^s\ (x, y) \end{gathered} \tag{2.2}$$

We will represent the quantities $u^{(2)}$, $v^{(2)}$, ..., $\tau_{yz}^{(2)}$, which define the second component of the stress state, in the form

$$\begin{gathered} u^{(2)} = \sum_s h^{s-2} u_2^s\ (u, v), \quad w^{(2)} = \sum_s h^{s-1} w_2^s \\ \sigma_x^{(2)} = \sum_s h^{s-2} \sigma_{x2}^s\ (x, y, z), \quad \tau_{xy}^{(2)} = \sum_s h^{s-2} \tau_{xy2}^s, \quad \tau_{xz}^{(2)} = \sum_s h^{s-1} \tau_{xz2}^s\ (x, y) \end{gathered} \tag{2.3}$$

In series (2.2) and (2.3) and everywhere henceforth $\sum_s$ denotes summation over the index s, which takes values of 1, 3, ..., $2n - 1$, where n is the number of the asymptotic approximation.

According to the chosen form of the asymptotic series, the overall solution (2.1) contains the zeroth and all positive powers of the parameter h. As $h \to 0$ the components of the displacement vector and the stress tensor will have a singularity $O(h^{-1})$. The latter is obvious for all the required quantities apart from the shear stress τ_{xy}, since the expression for $\tau_{xy}^{(1)}$ contains the term $h^{-2}\tau_{xy1}^1$. It will follow from the algorithm henceforth that $\tau_{xy1}^1 = 1$. The shear stresses τ_{xy} will therefore have the same singularity as all the other unknowns.

Since the procedure for constructing the recurrence processes for calculating $u_1^s, u_2^s, \ldots, \tau_{yz1}^s, \tau_{yz2}^s$ has been described in detail earlier,[1,2] we will only write the result here. We will calculate the required quantities in series (2.2) from the formulae

$$
\begin{aligned}
u_1^s &= p_x^s + \int_0^{\varsigma} (G_{xz}^{-1}\tau_{xz1}^s - \partial_x w_1^s)d\varsigma \ (x, y; u, \upsilon) \\
w_1^s &= f_z^s + \int_0^{\varsigma} \left(\frac{\sigma_{z1}^{s-2}}{E_z} - \frac{\nu_{zx}}{E_x}\sigma_{x1}^{s-2} - \frac{\nu_{zy}}{E_y}\sigma_{y1}^{s-2}\right)d\varsigma \\
\sigma_{x1}^s &= \frac{E_x}{1-\nu_{xy}\nu_{yx}}\left(\partial_x u_1^s + \nu_{xy}\partial_y \upsilon_1^s + \frac{\nu_{xz}+\nu_{yz}\nu_{xy}}{E_z}\sigma_{z1}^s\right) \ (x, y; u, \upsilon) \\
\sigma_{z1}^s &= p_z^s - \int_0^{\varsigma} (\partial_x\tau_{xz1}^s + \partial_y\tau_{yz1}^s)d\varsigma, \quad \tau_{xy1}^s = G_{xy}\gamma^{s-2} \\
\tau_{xz1}^s &= f_x^s - \int_0^{\varsigma} (G_{xy}\partial_y\gamma^{s-2} + \partial_x\sigma_{x1}^{s-2})d\varsigma \ (x, y) \\
\gamma^{s-2} &= \partial_x\upsilon_1^{s-2} + \partial_y u_1^{s-2}, \quad s = 1, 3, \dots, 2n-1
\end{aligned}
\tag{2.4}
$$

The functions f_x^s, f_y^s, f_z^s and p_x^s, p_y^s, p_z^s of the arguments x, y, which occur in the integrations, are arbitrary. The quantities with negative superscripts in formulae (2.4) must be put equal to zero.

The required quantities $u_2^s, \upsilon_2^s, \dots, \tau_{yz2}^2$ of the second components are found in the form

$$
\begin{aligned}
u_2^s &= g_x^s + \int_0^{\varsigma} (G_{xz}^{-1}\tau_{xz2}^{s-2} - \partial_x w_2^{s-2})d\varsigma \ (x, y; u, \upsilon) \\
w_2^s &= q_z^s + \int_0^{\varsigma} \left(\frac{\sigma_{z2}^s}{E_z} - \frac{\nu_{zx}}{E_x}\sigma_{x2}^s - \frac{\nu_{zy}}{E_y}\sigma_{y2}^s\right)d\varsigma \\
\sigma_{x2}^s &= \frac{E_x}{1-\nu_{xy}\nu_{yz}}\left(\partial_x u_2^s + \nu_{xy}\partial_y \upsilon_2^s + \frac{\nu_{xz}+\nu_{yz}\nu_{xy}}{E_z}\sigma_{z2}^s\right) \ (x, y; u, \upsilon) \\
\sigma_{z2}^s &= g_z^s - \int_0^{\varsigma} (\partial_x\tau_{xz2}^{s-2} + \partial_y\tau_{yz2}^{s-2})d\varsigma, \quad \tau_{xy2}^s = G_{xy}(\partial_y u_2^s + \partial_x \upsilon_2^s) \\
\tau_{xz2}^s &= q_x^s - \int_0^{\varsigma} (\partial_x\sigma_{x2}^s + \partial_y\tau_{xy2}^s)d\varsigma \ (x, y) \\
s &= 1, 3, \dots, 2n-1
\end{aligned}
\tag{2.5}
$$

The functions g_x^s, g_y^s, g_z^s and q_x^s, q_y^s, q_z^s of the arguments x and y, which occur in the integration, are arbitrary.

When constructing the model of the strain, the functions that occur in the integration in both the first and second recurrence processes must be determined while satisfying the conditions on the boundary of the slab. The first and subsequent approximations introduce twelve arbitrary functions $f_x^s, \dots, q_z^s$ into the overall solution (2.1). Hence, in the approximation with number n, we will have $12n$ functions of the coordinates x and y to satisfy the boundary conditions.

3. An explicit form of the solution from the asymptotic algorithm

The algorithm requires the successive calculation of all the parameters of the stress-strain state in the first, second and subsequent approximations. An analysis of formulae (2.2)–(2.5) enables this procedure to be simplified considerably.

For brevity all the calculations will henceforth be carried out solely for an isotropic slab with a modulus of elasticity E, a Poisson's ratio ν and a shear modulus $G = E/[2(1+\nu)]$. It is easy to see that in formulae (2.2) in each approximation all the quantities are defined using τ^s_{xz1}, τ^s_{yz1} and w^s_1. Hence, we can represent them by simple transformations as follows (the integration over z everywhere henceforth is carried out in the limits from 0 to z):

$$\tau^s_{xz1} = f^s_x - zT^{s-2}_{x1} + \iint R_{x1}(\tau^{s-2}_{xz1}, \tau^{s-2}_{yz1}, w^{s-2}_1)dz^2 \ (x, y)$$

$$w^s_1 = f^s_z + zT^{s-2}_{z1} + \iint R_{z1}(\tau^{s-2}_{xz1}, \tau^{s-2}_{yz1}, w^{s-2}_1)dz^2 \ (x, y)$$

Here we have used the following notation

$$T^{s-2}_{x1} = \frac{1}{h}\left(G\partial^2_y p^{s-2}_x + \frac{E}{1-\nu^2}\partial^2_x p^{s-2}_x + \frac{E}{2(1-\nu)}\partial^2_{xy} p^{s-2}_y + \frac{\nu}{1-\nu}\partial_x p^{s-2}_z\right) (x, y)$$

$$R_{x1}(\tau^{s-2}_{xz1}, \tau^{s-2}_{yz1}, w^{s-2}_1) = h^{-2}\left(\frac{\nu-2}{1-\nu}\partial^2_x\tau^{s-2}_{xz1} - \partial^2_y\tau^{s-2}_{xz1} + \frac{1}{\nu-1}\partial^2_{xy}\tau^{s-2}_{yz1} + \frac{E}{1-\nu^2}\partial_x\nabla^2 w^{s-2}_1\right) (x, y)$$

$$T^{s-2}_{z1} = \frac{1}{h}\left[\frac{1-\nu-2\nu^2}{E(1-\nu)}p^{s-2}_z + \frac{\nu}{\nu-1}(\partial_x p^{s-2}_x + \partial_y p^{s-2}_y)\right]$$

$$R_{z1}(\tau^{s-2}_{xz1}, \tau^{s-2}_{yz1}, w^{s-2}_1) = h^{-2}\left[\frac{1+\nu}{E(\nu-1)}(\partial_x\tau^{s-2}_{xz1} + \partial_y\tau^{s-2}_{yz1}) + \frac{\nu}{1-\nu}\nabla^2 w^{s-2}_1\right]$$

The relations obtained enable us to write expressions for the shear stresses and the bending in explicit form

$$\tau^{(1)}_{xz} = \sum_s h^{s-2}\sum_{k=0}^{s-1} A^{s,k}_x z^k \ (x, y), \quad w^{(1)} = \sum_s h^{s-2}\sum_{k=0}^{s-1} A^{s,k}_z z^k \tag{3.1}$$

The expansion coefficients $A^{s,k}_x$, $A^{s,k}_y$, $A^{s,k}_z$ are expressed in terms of arbitrary functions as follows:

$$A^{s,0}_x = f^s_x, \quad A^{s,1}_x = -T^{s-2}_{x1} \ (x, y), \quad A^{s,0}_z = f^s_z, \quad A^{s,1}_z = T^{s-2}_{z1}$$

$$A^{s,k}_x = \frac{1}{k(k-1)}R_{x1}(A^{s-2,k-2}_x, A^{s-2,k-2}_y, A^{s-2,k-2}_z) \ (x, y)$$

$$A^{s,k}_z = \frac{1}{k(k-1)}R_{z1}(A^{s-2,k-2}_x, A^{s-2,k-2}_y, A^{s-2,k-2}_z)$$

$$k = 2, 3, \dots, s-1; \quad s = 1, 3, \dots, 2n-1$$

Quantities with negative superscripts must be put equal to zero.

In the system of equations of the theory of elasticity, the first two equations of (1.1) and the third of equations (1.2) are approximately satisfied by algorithm (2.2), (2.4). The remaining six equations are exactly satisfied. Hence, for the known expressions (3.1) for the shear stresses and the bending, from these six equations we can determine the remaining unknown quantities. As a result we obtain

$$\begin{aligned} u^{(1)} &= P_x(x,y) + \int (G^{-1}\tau_{xz}^{(1)} - \partial_x w^{(1)})dz \ (u, \upsilon; x, y) \\ \sigma_x^{(1)} &= \frac{E}{1-\nu^2}(\partial_x u^{(1)} + \nu\partial_y \upsilon^{(1)}) + \frac{\nu}{1-\nu}\sigma_z^{(1)} \ (u, \upsilon; x, y) \\ \sigma_z^{(1)} &= P_z(x,y) - \int (\partial_x \tau_{xz}^{(1)} + \partial_y \tau_{yz}^{(1)})dz, \quad \tau_{xy}^{(1)} = G(\partial_y u^{(1)} + \partial_x \upsilon^{(1)}) \end{aligned} \tag{3.2}$$

Here we have put

$$P_x = \sum_s h^{s-1} p_x^s \ (x, y, z) \tag{3.3}$$

Hence, all the parameters of the first component of the stress-strain state can be calculated in terms of the shear stresses and bending of the slab. Hence, the algorithm for calculating these components can be conventionally called the problem of the shear and bending of a slab.

Similar transformations can be carried out for the second component. In algorithm (2.5) we will represent the expressions for u_2^s, υ_2^s, σ_{z2}^s in the form

$$\begin{aligned} u_2^s &= g_x^s + zT_{x2}^{s-2} - \iint R_{x2}(u_2^{s-2}, \upsilon_2^{s-2}, \sigma_{z2}^{s-2})dz^2 \ (u, \upsilon; x, y) \\ \sigma_{z2}^s &= g_z^s - zT_{z2}^{s-2} + \iint R_{z2}(u_2^{s-2}, \upsilon_2^{s-2}, \sigma_{z2}^{s-2})dz^2 \end{aligned}$$

Here we have put

$$T_{x2}^{s-2} = \frac{1}{h}(G^{-1}q_x^{s-2} - \partial_x q_z^{s-2}) \ (x, y), \quad T_{z2}^{s-2} = \frac{1}{h}(\partial_x q_x^{s-2} + \partial_y q_y^{s-2})$$

$$R_{x2}(u_2^{s-2}, \upsilon_2^{s-2}, \sigma_{z2}^{s-2}) = h^{-2}\left(\frac{2-\nu}{1-\nu}\partial_x^2 u_2^{s-2} + \partial_y^2 u_2^{s-2} + \frac{1}{1-\nu}\partial_{xy}^2 \upsilon_2^{s-2} + \frac{1+\nu}{E(1-\nu)}\partial_x \sigma_{z2}^{s-2}\right)$$

$(u, \upsilon; x, y)$

$$R_{x2}\left(u_2^{s-2} \sim \upsilon_2^{s-2}, \sigma_{z2}^{s-2}\right) = \frac{1}{h^2}\nabla^2\left[\frac{E}{1-\nu^2}\left(\partial_x u_2^{s-2} + \partial_y \upsilon_2^{s-2}\right) + \frac{\nu}{1-\nu}\sigma_{z2}^{s-2}\right]$$

These formulae enable us to write an explicit expression for the tangential displacements and the compression stresses. We have

$$u^{(2)} = \sum_s h^{s-2}\sum_{k=0}^{s-1} B_x^{s,k} z^k \ (u, \upsilon; x, y), \quad \sigma_z^{(2)} = \sum_s h^{s-2}\sum_{k=0}^{s-1} B_z^{s,k} z^k \tag{3.4}$$

Here we have used the following notation

$$B_x^{s,0} = g_x^s, \quad B_x^{s,1} = T_{x2}^{s-2} \ (x, y), \quad B_z^{s,0} = g_z^s, \quad B_z^{s,1} = -T_{z2}^{s-2}$$

$$B_x^{s,k} = -\frac{1}{k(k-1)}R_{x2}(B_x^{s-2,k-2}, B_y^{s-2,k-2}, B_z^{s-2,k-2}) \ (x, y)$$

$$B_z^{s,k} = \frac{1}{k(k-1)}R_{z2}(B_x^{s-2,k-2}, B_y^{s-2,k-2}, B_z^{s-2,k-2})$$

$$k = 2, 3, \ldots, s-1; \quad s = 1, 3, \ldots, 2n-1$$

The quantities with negative superscripts must be put equal to zero.

The second and third equations of (1.3) and the third of the equations (1.1) are approximately satisfied by algorithm (2.3), (2.5). The remaining six equations of the theory of elasticity are satisfied exactly. Hence, for the known expressions (3.4), from these six equations we can determine the remaining unknown quantities. As a result we obtain

$$\begin{aligned}\sigma_x^{(2)} &= \frac{E}{1-\nu^2}(\partial_x u^{(2)} + \nu\partial_y \upsilon^{(2)}) + \frac{\nu}{1-\nu}\sigma_z^{(2)}\ (u, \upsilon; x, y)\\ \tau_{xy}^{(2)} &= G(\partial_y u^{(2)} + \partial_x \upsilon^{(2)}),\quad \tau_{xz}^{(2)} = Q_x(x, y) - \int(\partial_x\sigma_x^{(2)} + \partial_y\tau_{xy}^{(2)})dz\ (x, y)\\ w^{(2)} &= Q_z(x, y) + \frac{1}{E}\int[\sigma_z^{(2)} - \nu(\sigma_x^{(2)} + \sigma_y^{(2)})]dz\end{aligned} \tag{3.5}$$

Here we have put

$$Q_x = \sum_s h^{s-1} q_x^s\ (x, y, z) \tag{3.6}$$

In the second component all the parameters of the stress-strain state can be calculated in terms of the tangential displacements and the compression stress. Hence, the process of determining this component can be conditionally called problem of extension compression in three directions.

4. The asymptotic algorithm and the power-series method

In the power-series method the solution of the system of equations (1.1)–(1.3) is expanded in power series with respect to the coordinate z. Previous calculations show that this method is close to the asymptotic integration method. However, when certain conditions are satisfied these methods become identical. We will write expressions (2.1) using relations (3.1)–(3.6). As a result we obtain the following representation of the parameters of the stress-strain state

$$u = \sum_{k=0}^{m} u_k(x, y)z^k,\quad \upsilon = \sum_{k=0}^{m} \upsilon_k(x, y)z^k, \ldots,\quad \tau_{yz} = \sum_{k=0}^{m}\tau_{yzk}(x, y)z^k \tag{4.1}$$

The expansion coefficients u_k, υ_k,... τ_{yzk} in these series will be expressed in terms of the functions f_x^s, f_y^s, ... q_z^s.

Relations (4.1) show that the asymptotic algorithm can be regarded as a method of calculating the expansion coefficients in the power-series method. The series obtained in the general case will contain the zeroth and all positive powers of the coordinate z. The maximum power m in expansions (4.1) will be the same for all the components of the displacements and stresses: in the first approximation $m = 1$, in the second $m = 3$, and for the approximation with number n the maximum exponent $m = 2n - 1$.

5. The asymptotic algorithm and the method of hypotheses

The basis of the method of hypotheses is certain assumptions regarding the nature of the stress-strain state, which, using Eqs. (1.1)–(1.3), later lead to one or other models of the theory of plates and shells. The equations of the theory of elasticity enable us to realize this method in different forms.[4,5]

An analysis of the transformations of the system of equations (1.1)–(1.3) for asymptotic integration enables us to propose the following version of the method of hypotheses. The solution of the equations of the theory of elasticity, as before, will be constructed using formulae (2.1). To determine the first component in the n-th approximation we will specify the expressions of the shear stresses and bending in the form

$$\tau_{xz}^{(1)} = \sum_{k=0}^{2n-2} F_x^k(x, y)z^k\ (x, y),\quad w^{(1)} = \sum_{k=0}^{2n-2} F_z^k(x, y)z^k \tag{5.1}$$

The remaining required quantities are found from formulae (3.2).

We will compare the coefficients of like powers of the coordinate z in series (3.1) and (5.1). For these series to be identical it is necessary that the arbitrary functions of the method of hypotheses should be expressed in terms of arbitrary functions of the asymptotic method using the following formulae

$$F_x^k = \begin{cases} \sum\limits_{s=k,k+2,\ldots}^{2n-2} h^{s-1} A_x^{s+1,k}, & k = 0, 2, \ldots, 2n-2\ (x, y, z) \\ \sum\limits_{s=k,k+2,\ldots}^{2n-3} h^s A_x^{s+2,k}, & k = 1, 3, \ldots, 2n-3\ (x, y, z) \end{cases} \tag{5.2}$$

When these relations are satisfied the shear stresses $\tau_{xz}^{(1)}$, $\tau_{yz}^{(1)}$ and the bending $w^{(1)}$, calculated by the method of hypotheses and from the asymptotic expansion, will be identically equal. For the remaining parameters of the first component of the stress-strain state to be equal formulae (3.3) must be satisfied.

When determining the second component of the stress-strain state of the slab, the following algorithm is used to realize the method of hypotheses. We initially specify the expressions of the tangential displacements and normal compression stresses

$$u^{(2)} = \sum_{k=0}^{2n-2} G_x^k(x,y) z^k\ (u, v; x, y), \quad \sigma_z^{(2)} = \sum_{k=0}^{2n-2} G_z^k(x,y) z^k \tag{5.3}$$

Comparing series (3.4) and (5.3) we conclude that the tangential displacements and the compression stresses in the methods considered will be identical when conditions identical to conditions (5.2) are satisfied with F and A replaced by G and B.

The remaining unknowns are determined from relations (3.5). For these to be identical in both methods, formulae (3.6) must be satisfied.

Hence, in the version of the method of hypotheses proposed for constructing the approximation with number n, as in the asymptotic algorithm we have $12n$ arbitrary functions F_x^k, G_x^k, P_x, $Q_x, \ldots, Q_z$. In any approximation, the forms of the power series are completely identical and when the above-mentioned conditions are satisfied, both methods lead to the same strain models.

6. The asymptotic algorithm and the method of successive approximations

The algorithms derived can be realized in the form of the method of successive approximations. When constructing series (3.1) to calculate the first component in the zeroth approximation we will assume that the normal stresses $\sigma_x^{(1)}$, $\sigma_y^{(1)}$, $\sigma_z^{(1)}$ and the shear stresses $\tau_{xy}^{(1)}$ are equal to zero. Then, from the first two equations of (1.1) and the third equation of (1.2) we obtain in the first approximation

$$\tau_{xz}^{(1,1)} = \Phi_x^1(x,y), \quad \tau_{yz}^{(1,1)} = \Phi_y^1(x,y), \quad w^{(1,1)} = \Phi_z^1(x,y)$$

The remaining unknowns are calculated from formulae (3.2). We will denote the functions of the x and y coordinates which occur in the integration by $\prod_x^1$, $\prod_y^1$, $\prod_z^1$. The calculation of the second approximation begins with the integration of the first two equations of (1.1) and the third equation of (1.2). From these we obtain $\tau_{xz}^{(1,2)}$, $\tau_{yz}^{(1,2)}$, $w^{(1,2)}$. We then use formulae (3.2) and as a result new functions $\prod_x^2$, $\prod_y^2$, $\prod_z^2$ etc. appear. Hence, the transfer from one approximation to another is made by means of three relations

$$\tau_{xz}^{(1,k)} = \Phi_x^k - \int(\partial_x \sigma_x^{(1,k-1)} + \partial_y \tau_{xy}^{(1,k-1)}) dz\ (x, y)$$

$$w^{(1,k)} = \Phi_z^k + \frac{1}{E}\int[\sigma_z^{(1,k-1)} - \nu(\sigma_x^{(1,k-1)} + \sigma_y^{(1,k-1)})] dz$$

$$k = 1, 2, \ldots, n$$

We will determine the second component of the stress-strain state by assuming that, in the zeroth approximation, there are no shear stresses $\tau_{xz}^{(2)}$, $\tau_{yz}^{(2)}$ or bending $w^{(2)}$ in the slab. Then, from the last two equations of (1.3) and the last equation of (1.1) we obtain, in the first approximation, expressions for the tangential displacements and compression stresses

$$u^{(2,1)} = \Gamma_x^1(x,y), \quad \upsilon^{(2,1)} = \Gamma_y^1(x,y), \quad \sigma_z^{(2,1)} = \Gamma_z^1(x,y)$$

Relations (3.5) enable us to calculate the remaining unknowns. We will denote the arbitrary functions of the coordinates of the middle plane which then appear by $K_x^1,\ K_y^1,\ K_z^1$. We transfer to the second approximation by means of the last two equations of (1.3) and the last equation of (1.1). When calculating $u^{(2,2)}$, $\upsilon^{(2,2)}$, $\sigma_z^{(2,2)}$ we use relations (3.5), and functions $K_x^2,\ K_y^2,\ K_z^2$ etc. appear. We transfer to the next approximation by means of the formulae

$$u^{(2,k)} = \Gamma_x^k + \int(G^{-1}\tau_{xz}^{(2,k-1)} - \partial_x w^{(2,k-1)})dz \ (u, \upsilon; x, y)$$

$$\sigma_z^{(2,k)} = \Gamma_z^k - \int(\partial_x\tau_{xz}^{(2,k-1)} + \partial_y\tau_{yz}^{(2,k-1)})dz$$

$$k = 1, 2, \ldots, n$$

Summing the results of the two processes, we obtain the solution of the system of equations of the theory of elasticity in the form (4.1), where the coefficients $u_k, \upsilon_k, \ldots, \tau_{yzk}$ in approximation with number n will be expressed by means of $12n$ arbitrary functions $\Phi_x^k,\ \prod_x^k,\ \Gamma_x^k,\ K_x^k,\ \Phi_y^k,\ \ldots,\ K_z^k$.

Comparing this algorithm with the algorithm of the asymptotic method, we find that these two methods are interconnected. The results will be identical if the following relations are satisfied in the approximation with number n

$$\Phi_x^k = \sum_s^k h^{s-2} \overset{s}{f}_x, \quad \Gamma_x^k = \sum_s^k h^{s-2} \overset{s}{g}_x, \quad \Pi_x^k = \sum_s^k h^{s-1} \overset{s}{p}_x, \quad K_x^k = \sum_s^k h^{s-1} \overset{s}{q}_x\ (x, y, z),$$

$$k = 1, 2, \ldots, 2n-1$$

Here $\sum_s^k$ denotes summation with respect to the index s, which takes values of $1, 3, \ldots, 2k-1$.

Hence, the algorithm of the method of successive approximations corresponds completely to the asymptotic algorithm.

7. Classification of the models of the strain of an elastic layer

The theoretical models of the strain of plates and slabs, of load-carrying binding layers in three-layer plates and of adhesive compounds result from the algorithms constructed for a specific choice of the form of the arbitrary functions. We have shown here that the different methods of solving the problem of the theory of elasticity lead to the same power series for all the components of the stress-strain state. This enables us to introduce a non-contradictory classification of the strain models, the basis of which is the maximum power of the coordinate z in series (4.1).

The calculation schemes which follow exactly from the structure of the first asymptotic approximation will be called models of the first approximation. It was proved in Ref. 6 that the Kirchhoff model is a scheme of the first approximation. The model for calculating the binding layers in multilayered structures[7] also belongs to the first approximation.

The calculation schemes of the second approximation must contain polynomials with a highest power with respect to the z coordinate of three. These include the model of the strain of orthotropic plates, in which the interlayer shear is given by a quadratic parabolic law with respect to the thickness,[5] the model of the strain of a filler in three-layer structures[8] and many other models.

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